27 de noviembre.

Lema: Sean $A \in M_{m \times n}(F)$ y $B \in M_{q \times m}(F)$.

tenemos las transformaciones lineales.

$$F^n \xrightarrow{L_A} F^m \xrightarrow{L_B} F^q \qquad F^n \xrightarrow{L_B \circ L_A} F^q$$

$$L_A(\bar{x}) = A\bar{x} \qquad L_B(\bar{y}) = B\bar{y}. \qquad (L_B \circ L_A)(\bar{x}) = \underbrace{B(A\bar{x})}.$$

también tenemos la matriz $\underset{q \times m}{B}\,\underset{m \times n}{A} \in M_{q \times n}(F)$.

$$L_{BA} : F^n \longrightarrow F^q. \qquad L_{BA}(\bar{x}) = \underbrace{(BA)\bar{x}}.$$

ENTONCES $L_B \circ L_A = L_{BA}$

1) Ejemplo.

Sean. $\overset{\alpha}{\mathbb{R}^3} \xrightarrow{T} \overset{\beta}{\mathbb{R}} \xrightarrow{G} \overset{\gamma}{\mathbb{R}^2}$ t. lineales

$T(x,y,z) = x$, $G(x) = (x,0)$.

Sean α, β, γ las bases canónicas ordenadas de $\mathbb{R}^3$, $\mathbb{R}$ y $\mathbb{R}^2$ respectivamente, con el orden usual.

Calcular $[G \circ T]_{\alpha}^{\gamma}$

Para ello hay que calcular los vectores de coordenadas.

$$(G \circ T)(1,0,0) = G(T(1,0,0)) = G(1) = (1,0) = 1(1,0) + 0(0,1).$$

$$(G \circ T)(0,1,0) = G(T(0,1,0)) = G(0) = (0,0) = 0(1,0) + 0(0,1).$$

$$(G \circ T)(0,0,1) = G(T(0,0,1)) = G(0) = (0,0) = 0(1,0) + 0(0,1).$$

$$[G \circ T]_{\alpha}^{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\beta = \{1\}$.

Ahora calculamos. $[T]_\alpha^\beta$ y $\underbrace{[G]_\beta^\gamma}$.

$T(1,0,0) = 1 = 1\,(1)$.

$T(0,1,0) = 0 = 0\,(1)$.

$T(0,0,1) = 0 = 0\,(1)$.

$[T]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$.

$G(1) = (1,0) = 1\,(1,0) + 0\,(0,1)$. $[G]_\beta^\gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$[G]_\beta^\gamma\,[T]_\alpha^\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [G \circ T]_\alpha^\gamma$$

2) Sean $\underset{\alpha}{V} \xrightarrow{T} \underset{\beta}{W} \xrightarrow{G} \underset{\gamma}{Z}$ t. lineales.

Sup. que $\dim V = n$, $\dim W = m$, $\dim Z = q$.

Sean $\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$ y $\gamma = \{z_1, \dots, z_q\}$ bases ordenadas de V, W y Z respectivamente.

Entonces

$$[\underbrace{G \circ T}]_{\alpha}^{\gamma} = [G]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}.$$

dem:

Recordemos que para cada $v \in V$.

$$[T]_\alpha^\beta [v]_\alpha = [T(v)]_\beta .$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_\alpha"'] \arrow[dr, phantom, "/\!/"] & W \arrow[d, "\varphi_\beta"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

$A = [T]_\alpha^\beta$.

Para cada $w \in W$,

$$[G]_\beta^\gamma [w]_\beta = [G(w)]_\gamma .$$

$$\begin{tikzcd} W \arrow[r, "G"] \arrow[d, "\varphi_\beta"'] \arrow[dr, phantom, "/\!/"] & Z \arrow[d, "\varphi_\gamma"] \\ F^n \arrow[r, "L_B"'] & F^m \end{tikzcd}$$

$B = [G]_\beta^\gamma$

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[rr, bend left, "G \circ T"] \arrow[d, "\varphi_\alpha"'] \arrow[dr, phantom, "//"] & W \arrow[r, "G"] \arrow[d, "\varphi_\beta"] \arrow[dr, phantom, "//"] & Z \arrow[d, "\varphi_\gamma"] \\
F^n \arrow[r, "L_A"'] \arrow[rr, bend right] & F^m \arrow[r, "L_B"'] & F^q
\end{tikzcd}
$$

$$L_B \circ L_A = \underbrace{L_{BA}}$$

$A = [T]_\alpha^\beta \qquad B = [G]_\beta^\gamma$.

Calculamos : $v \in V$.

$$\varphi_\gamma (G \circ T)(v) = \varphi_\gamma (G(T(v))).$$

$$= \underbrace{[G(T(v))]}_{}{}_\gamma$$

$$L_{BA} \circ \varphi_\alpha (v) = L_{BA}(\varphi_\alpha(v)).$$

$$= L_{BA}([v]_\alpha) = (BA)[v]_\alpha.$$

$$= B(\underbrace{A[v]_\alpha}) = B(\underbrace{[T(v)]_\beta}) = \underbrace{[G(T(v))]}_{}{}_\gamma \qquad //.$$

3) Ejemplo.

$$\overset{\alpha}{\mathbb{R}^2} \xrightarrow{\ T\ } \overset{\alpha}{\mathbb{R}^2} \xrightarrow{\ G\ } \overset{\alpha}{\mathbb{R}^2}.$$

$T(x,y) = (y,x) \qquad G(x,y) = (2x, 2y).$

elejimos $\alpha = \{(1,0), (0,1)\}$ base ordenada de $\mathbb{R}^2$.

Calcular $[T]_\alpha^\alpha$.

$T(1,0) = (0,1) = 0(1,0) + 1(0,1).$

$T(0,1) = (1,0) = 1(1,0) + 0(0,1).$

$$[T]_\alpha^\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calcular $[G]_{\alpha}^{\alpha}$.

$G(1,0) = (2,0) = 2(1,0) + 0(0,1)$.

$G(0,1) = (0,2) = 0(1,0) + 2(0,1)$

$[G]_{\alpha}^{\alpha} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

$[G \circ T]_{\alpha}^{\alpha} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

$T : \underset{\alpha}{V} \longrightarrow \underset{\alpha}{V}$. $[T]_{\alpha}^{\alpha} = [T]_{\alpha}$

4) Sean V y W F-e.v.

$\dim V = n$, $\dim W = m$.

Entunces hay un isomorfismo.

$$\begin{array}{c} \mathrm{Hom}_F(V,W) \cong M_{m\times n}(F). \\ \parallel \\ \{T: V \to W \mid T \text{ es t. lineal}\} \end{array}$$

$$\mathrm{Hom}_F(V,W) \longrightarrow M_{m\times n}(F).$$

Sea $\underbrace{T : \overset{n}{V} \longrightarrow \overset{m}{W}}$ una t. lineal

Sean α y β bases ordenadas de V y W respect.

Definimos. $f : \mathrm{Hom}_F(V,W) \longrightarrow M_{m\times n}(F).$

$$f(T) = [T]_\alpha^\beta.$$

Si $\dim V = \dim W$, ent T es ISO $\Leftrightarrow$ $\underbrace{[T]_\alpha^\beta}$ es invertible.

T^{-1}